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On an Immediate Generalization of Local Theorems in which the Generating Point divides a Variable Linear Segment in a Constant Ratio.

By Samuel Roberts,

President of the London Mathematical Society.

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1. In what follows I make use of the obvious principle, that, if two curves have a one to one correspondence, and if the points on a particular straight line of one correspond to points on a straight line of the other, the curves are of the same order and deficiency. The particular straight line is, in the cases I shall discuss, the line at infinity for both curves.

2. A curve may be determined as the locus of a point, which divides in a constant ratio a terminated straight line variable in length and position; and a family of curves related to one another is obtained by changing the ratio. Each such family depends, therefore, on one parameter. An additional parameter is introduced by the transformation of which I propose to treat.

There are commonly particular values of the ratio, which give curves of a lower order than that of the general locus, especially by the reduplication of the locus or a curve-factor of the locus.

For example, if straight lines through a point meet a circle, the locus of the middle points of the intercepts by the circle is another circle through the fixed point, and the centre of the direction.

This is plainly a special locus, and if the chords are divided by the generating point in the ratio $k : l$, the locus is found to be the inverse of a conic, i. e. of the fourth order. We may look for a similar reduction of the order whenever the ratio is one of equality, and the extremities of the linear segments move on one and the same curve.

It is to be understood that when I hereafter speak of a curve of the class described, which for brevity I shall call a ratio-curve, I mean one general as to order; in fact, I suppose the ratio of division to be denoted by general literal symbols, to which such values may be ascribed as will not give rise to a special reduction of order.

3. I propose to establish, that, if similar triangles, uniformly directed with respect to the generating segments, be superposed thereon, each having for base the corresponding segment, the general locus of the vertex, say, the vertex-curve, is of the same order and the same deficiency as the corresponding ratio-curve.

This is known to be true, for instance, of loci so derived from a constant line moving in plane space. The triangles are then all equal, and the result is, that the general locus of a point rigidly connected with the moving line is of the same order and deficiency as the corresponding ratio-curve, — a conclusion of some kinematical importance.

The statement is also manifestly true with respect to a curve referred to polar co-ordinates. If we divide the radii vectores in a given ratio, a similar and similarly placed curve is described by the dividing point; and if triangles are constructed on these vectores in the manner proposed, we have a similar curve turned through an angle.

And first of all, as to the deficiency, the correspondence of the ratio-curve and the vertex-curve is, by the nature of the construction, one to one, so that their deficiency is the same. The complete ratio and vertex curves may, however, break up into corresponding factors, and if so, the deficiencies of the corresponding factors are the same. It is the question of order which requires special consideration.

4. Let AB be the linear segment in one of its positions, divided at the point P in a given ratio. Erect QP perpendicular to AB. The triangle AQB will remain similar and uniformly directed if

$$AP : BP : PQ = k : l : p,$$

where k, l, p are constant and finite quantities.

If, therefore, X, Y, are the co-ordinates of Q; x_1, y_1, and x_2, y_2 those of A and B, respectively; and θ is the angle which AB makes with a fixed line, the axis of x, we have

$$\begin{aligned} X &= x_1 + m\,(k\cos\theta - p\sin\theta) \\ Y &= y_1 + m\,(k\sin\theta + p\cos\theta) \\ X &= x_2 + m\,(l\cos\theta - p\sin\theta) \\ Y &= y_2 + m\,(l\sin\theta + p\cos\theta), \end{aligned} \qquad \text{(A)}$$

where m and θ are variable parameters.

In order to define the locus, we must have further conditions equivalent to three independent conditions in $x_1, y_1, x_2, y_2, m, \theta$. These may be taken to be independent of k, l, p; for if these constants should be included among the constants of the complementary conditions, we are at liberty to change p, k, l in (A) into p', k', l'.

Now, although by giving form to such further conditions, we can represent cases of great generality, the complete proposition cannot be established by this means. We must obtain our conclusion independently of the unexpressed conditions.

The system of equations corresponding to (A), but belonging to the ratio-curve described by the foot of the perpendicular QP, is

$$\begin{aligned} X' &= x_1 + mk \cos \theta \\ Y' &= y_1 + mk \sin \theta \\ X' &= x_2 + ml \cos \theta \\ Y' &= y_2 + ml \sin \theta. \end{aligned} \tag{B}$$

By combining (A) and (B), we may obtain a variety of relations. It will be sufficient to write down

$$\begin{aligned} (l-k)\, X' &= lx_1 - kx_2 \\ (l-k)\, Y' &= ly_1 - ky_2 \end{aligned} \tag{a}$$

$$\begin{aligned} X &= X' - \tfrac{p}{k}(Y' - y_1) & \qquad X &= X' - \tfrac{p}{l}(Y' - y_2) \\ Y &= Y' + \tfrac{p}{k}(X' - x_1) & \qquad Y &= Y' + \tfrac{p}{l}(X' - x_2). \end{aligned} \tag{b}$$

5. Now, if we write for x_1, $\rho_1(\cos\alpha + i\sin\alpha)$, for y_1, $r_1(\cos\beta + i\sin\beta)$, etc., where α, β, etc., are real angles and ρ_1, r_1, etc., are real and positive moduli, we see, from the foregoing expressions, that, if the moduli of (x_1, y_1), (x_2, y_2) are finite, then those of (X', Y') and (X, Y) are also finite. And if the moduli of (x_1, y_1) or of (x_2, y_2) are infinite, or one of them is infinite, but the moduli of the remaining co-ordinates (x_2, y_2) or (x_1, y_1) are finite, then the moduli of (X', Y') and (X, Y), or one of each is infinite.*

No doubt we ought to be able to derive all the conclusions we want from the same systems (A) and (B), when the segment lies altogether at infinity. There are, however, obvious difficulties in the way, especially when we consider the circular asymptotes. These are "lines of no length." Nevertheless, the co-ordinates of points on a circular asymptote have determinate moduli, so that part of the difficulty may be evaded; but, again, a circular asymptote makes any real angle with itself. It is more satisfactory to employ another process than to argue directly on infinites and the circular points.

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* This is so stated to cover the case of a point at infinity, determined by a line parallel to an axis, when one of the co-ordinates is finite.

6. By the very nature of the construction the correspondence of (X', Y') and (X, Y) depends on the position and length of the corresponding segment alone, and to determine it, we are at liberty to arrive at any proposed position of the segment as we conveniently can. Moreover, the actual length of the segment is immaterial when we have only to distinguish between finite and infinitely distant points. This applies to imaginary as well as real positions. These considerations enable us to have recourse to certain elementary cases; and to avoid direct reasoning on isolated infinite quantities, I take, therefore, some essential cases in order.

I. *One extremity of the segment finite and real, the other at infinity and real.*

Let A be the finite extremity. In the direction of the segment take AB finite, and construct the triangle AQB and the perpendicular QP, as before.

If now we suppose B to move to infinity, the triangle AQB remaining of the same angularity, P and Q ultimately lie at infinity, P coinciding with B, and Q being determined by the direction AQ.

This agrees with the result of neglecting x_2, y_2 in (a) and (b).

II. *The segment real and altogether at infinity.*

Take the lines OA, OB met by the finite segment AB, and construct the triangle AQB and the perpendicular QP, as before.

When AB moves to infinity, parallel to itself, P and Q also move to infinity, when the angularity of AQB is unchanged.

Since, however, the direction of AB is arbitrary, every point at infinity in turn may be made to represent the ultimate position of P, indicating that the line at infinity becomes an extraneous factor of the locus, the order of which is consequently reduced. But if we determine the direction of P, that of Q is also determined for a given angularity. The point to be noticed is that the mutually corresponding points, whether proper or extraneous, all lie at infinity.

The conclusions (I.) and (II.) will hold for imaginary positions of the segment when we have regard to the moduli of the co-ordinates, except that the case in which the segment coincides with a circular asymptote requires special determination.

7. III. Therefore, consider the elementary case of straight lines through a point and meeting two given straight lines. The ratio and vertex curves are conics, passing through the points at infinity, corresponding respectively to those of the directrices. And since in this case all finite points (X', Y') correspond to finite points (X, Y), and *vice versa*, it follows, generally, that this result holds for finite positions of the segment, whether imaginary or real, even when the segment forms part of a circular asymptote. But since this is so, and the correspondence of the ratio-curve to the vertex-curve is one to one, it follows that to infinitely distant points on the ratio-curve must correspond infinitely distant points on the vertex-curve. Consequently the order must be the same for both.

For the purpose of showing the effect when the one extremity of the segment coincides with a circular point at infinity, it is more convenient to take the case of a point and circle.

The ratio-curve and the vertex-curve are both of them circles, and the corresponding points (X', Y') (X, Y) coincide at the circular points.

IV. We must consider the case in which the segment passes through both the circular points, and therefore coincides with the line at infinity.

If parallel lines meet a straight line and a circle, the ratio and vertex curves are conics, and the infinitely distant points of one conic correspond to those, respectively, of the other.

According to the same principles, the effect may be shown of the coincidence of two extremities of a segment. Thus, if a pencil of parallel lines meets a parabola, the ratio-curve and the vertex-curve are both parabolas, and the points of contact with the line at infinity correspond. It is not necessary, however, to pursue this further for the end in view.

We see, then, that to points at infinity on the ratio-curve always correspond points at infinity on the vertex-curve; and since the correspondence is one to one, the order must be the same for both.

Since the vertex-curve can be obtained by taking the ratio-curve as one of the directrices, to an extraneous factor of the one will correspond an extraneous factor of the other; and if the one curve is composite, so is the other in a corresponding manner.

It is to be observed that such composite loci may still be reckoned as proper loci, in our present point of view.

8. The conclusion at which we have arrived is of considerable utility for the extension of problems on loci. In many cases a ratio-curve can be determined by simple processes, when the analytical investigation of the corresponding vertex-curve involves complicated work.

Take, for instance, a pencil of straight lines meeting two directrices. In the ratio-curve, adopting polar co-ordinates with the vertex of the pencil as pole, we have to eliminate ρ_1, ρ_2 from

$$R = m\rho_1 + n\rho_2$$

$$\phi(\rho_1, \theta) = 0 \qquad \psi(\rho_2, \theta) = 0.$$

This is often a simple matter where the corresponding elimination for the vertex-curve is lengthy or impracticable.

Suppose the directrices are two circles, the ratio-curve is readily found to be of the sixth order (tricircular), with three other nodes, reducing to the fourth order (the inverse of a conic) when the pole is on one of the circles, and to the second order (a circle) when the pole is one of the finite intersections of the directrices. It is tedious, though of course practicable, to show analytically that the vertex-curve is also in the first case a sextic of similar kind, in the second the inverse of a conic, and in the third a circle. The last result is, however, well known in connection with the theory of three circles intersecting in a point.

9. Again, it will also be observed that we can project orthogonally a figure relative to a ratio-curve, and reproduce a ratio-curve together with its appropriate conditions of generation. The vertex-curve cannot be so projected, and consequently we get an entirely new theorem.

For example, projecting the last case, above given, we have the following:—

If a vector be drawn through a finite intersection of two similar and similarly placed conics, the locus of a point dividing the intercepts in a constant ratio is a similar and similarly placed conic passing through the finite intersections together, strictly speaking, with two similar and similarly placed conics obtained by measuring the intercepts from the pole.

We have for the vertex-curve the following:—

If a vector be drawn through a finite intersection of two similar and similarly placed conics, the vertices of similar and similarly directed triangles on the intercepts as base describe a conic through the other finite intersection of the two given conics (a conic not necessarily similar or similarly placed), together with two supplementary conics. This result is not to be derived by projection, since the triangles would not be projected into similar triangles.

10. It has served our purpose to consider examples belonging to the very simple case of lines enveloping a point. To illustrate different conditions, suppose that straight lines touch a given circle and meet a given straight line; the

locus of a point dividing the intercepts on the tangents between the point of contact and the line is a curve of the fourth order. In this case we get

$$X = x_1 + m\,(k \cos\theta - p \sin\theta)$$

$$Y = y_1 + m\,(k \sin\theta + p \cos\theta)$$

$$y_1 - \delta = 0$$

$$X = x_2 + m\,(l \cos\theta - p \sin\theta)$$

$$Y = y_2 + m\,(l \sin\theta + p \cos\theta)$$

$$x_2 \cos\theta + y_2 \sin\theta = 0$$

$$x_2^2 + y_2^2 = r^2,$$

whence we have

$$X \cos\theta + Y \sin\theta = ml$$

$$X \sin\theta - Y \cos\theta = -\,mp \pm r$$

$$Y - \delta = m\,(k \sin\theta + p \cos\theta).$$

Simple as these equations are, the reduction is troublesome; but if $p = 0$, then

$$X^2 + Y^2 = m^2 l^2 + r^2$$

$$mkX \sin\theta = X\,(Y - \delta)$$

$$mkX \cos\theta = m^2 kl - Y\,(Y - \delta),$$

giving

$$\frac{k^2}{l^2} X^2 (X^2 + Y^2 - r^2) - X^2 (Y - \delta)^2 - \left(\frac{k}{l}(X^2 + Y^2 - r^2) - Y(Y - \delta)\right)^2 = 0.$$

The curve is circular, having a double point at infinity determined by $Y = 0$, in accordance with our previous conclusions.

The vertex-curve is, therefore, also circular, of the fourth order, having a double point at infinity determined by $Xp - Yk = 0$.

We can now project the circle and line directrices into a central conic and line, and deduce a similar result for the vertex-curve in this case.

11. I have only further to remark, that the transformation of the ratio-curve into the vertex-curve is a particular case of the general transformation where the powers and products of X, Y are decomposed into their elements, and each X and Y receives an independent linear transformation.

We see this directly by writing (A) in the form

$$Xk + Yp = x_1k + y_1p + m\,(p^2 + k^2)\cos\theta = X_1 + m\,(p^2 + k^2)\cos\theta$$

$$Yk - Xp = y_1k - x_1p + m\,(p^2 + k^2)\sin\theta = Y_1 + m\,(p^2 + k^2)\sin\theta$$

$$Xl + Yp = x_2l + y_2p + m\,(p^2 + l^2)\cos\theta = X_2 + m\,(p^2 + l^2)\cos\theta$$

$$Yl - Xp = y_2l - x_2p + m\,(p^2 + l^2)\sin\theta = Y_2 + m\,(p^2 + l^2)\sin\theta$$

and

$$x_1 = \frac{X_1k - Y_1p}{p^2 + k^2} \qquad y_1 = \frac{X_1p + Y_1k}{p^2 + k^2}$$

$$x_2 = \frac{X_2l - Y_2p}{p^2 + l^2} \qquad y_2 = \frac{X_2p + Y_2l}{p^2 + l^2},$$

and by comparing these with (B).

Since, however, in this way of putting the matter, the unexpressed conditions vary with p, k, l, we cannot at once infer the equality of the orders of the ratio and vertex curves, for which purpose it has been necessary to enter into somewhat tedious details. I have not been able to hit upon a simpler proof, though it is highly probable such a one can be found. I thought it desirable to introduce the analytical expressions of Article 4, in order to show where the difficulty arises, but in fact we found them insufficient, and it would favor uniformity to adhere to the method of Articles 6 and 7.